In Memory of Victor Nikolaevich Popov

GAUSS DECOMPOSITION FOR QUANTUM GROUPS AND SUPERGROUPS

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Abstract.

The Gauss decompositions of the quantum groups, related to classical Lie groups and supergroups are considered by the elementary algebraic and R-matrix methods. The commutation relations between new basis generators (which are introduced by the decomposition) are described in some details. It is shown that the reduction of the independent generator number in the new basis to the dimension of related classical (super) group is possible. The classical expression for (super) determinant through the Gauss decomposition generators is not changed in the deformed case. The symplectic quantum group $Sp_q(2)$ and supergroups $GL_q(1|1), GL_q(2|1)$ are considered as examples.

[1] Supported by Russian Foundation for Fundamental Research, Grant N 94-01-01157-a.

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1. Introduction. The development of the quantum inverse scattering method (QISM) [1] intended for investigation of the integrable models of the quantum field theory and statistical physics naturally gives rise to some interesting algebraic constructions. Their investigation allows to select a special class of Hopf algebras now known as quantum groups and quantum algebras [2,3]. In many respects these algebraic structures resemble very much those of the standard Lie groups and Lie algebras, that is probably one of the reasons explaining the appearance of quite a few number of articles dealing with these objects. The nice R-matrix formulation of the quantum group theory [4], based on the fundamental relation of the QISM (the FRT-relation) has given an additional impulse for these investigations. The FRT-relation

$$RT_1T_2 = T_2T_1R, \tag{1}$$

may be considered as a condensed matrix form of homogeneous quadratic (commutation) relations between n^2 generators t_{ij}, $(i,j = 1,2,...,n)$ of the associative algebra F_q. The entries of the number $n^2 \times n^2$ matrix R play the role of the Lie algebra structure constants. We are using the QISM standard notations $T_1 = T \otimes I$, $T_2 = I \otimes T$, where $T = (t_{ij})$ is the $n \times n$-matrix of generators for the associative algebra F_q (q-**matrix**). Usually it is supposed that the R-matrix is a solution of the Yang-Baxter equation [4]. The R-matrices related to the classical simple Lie algebras from the Cartan list are well known [3-5]. The matrix form of the FRT-relation (1) allows to introduce without difficulty into the associative algebra F_q, together with multiplication, additional maps (coproduct, counit and antipod (provided that the inverse matrix T^{-1} exists)) and convert F_q into the Hopf algebra i.e. the **quantum group** [4]. Rather straightforward (and sometimes misleading) analogy of the q-matrix T with the standard matrices as elements of the classical matrix groups nevertheless allows one to consider many questions of the higher algebra without difficulties operating with quantum analogs of the well known classical objects such as determinants, minors, linear spaces, symmetric and external products etc.[4, 6-11].

In this paper we consider the Gauss decomposition of the q-matrix $T = T_LT_DT_U$ on strictly lower and upper triangular matrices (with unities on the diagonal) and the diagonal matrix $T_D := A$ using the elementary algebra methods and the R-matrix approach [4]. Such a decomposition of a given matrix (with non commuting entries) into the products of matrices of the special type (similar to the Gauss decomposition) is the particular case of the general factorization problem [12], which can be considered as a cornerstone of many constructions of the classical as well as quantum inverse scattering methods. It is worthwhile to point

out that in different contexts these decompositions have been appeared (sometimes in non explicit form) in many papers on the quantum deformation [8,12-29,74]. Let us comment, without details, some of them.

In a quite general framework of the quantum double construction the Gauss decomposition was introduced in the paper [13], where the universal triangular objects $\mathcal{M}^{\pm}$ were defined. Their matrix representations on one of the factors $(\rho \otimes 1)\mathcal{M}^{\pm} = M^{\pm}$ are the solutions of the FRT-relations

$$RM_1^{\pm}M_2^{\pm} = M_2^{\pm}M_1^{\pm}R, \quad M_1^{+}M_2^{-} = M_2^{-}M_1^{+}.$$

with the R-matrix $R = (\rho \otimes \rho)\mathcal{R}$ related to the universal one $\mathcal{R}$. Their product $M^{-}M^{+}$ after the unification of diagonal elements $(M_{ii}^{-}M_{ii}^{+} = A_{ii})$ gives the $q-$matrix T. Usually the new generators, which are defined by the Gauss decomposition, have simpler commutation rules (multiplication) but more complicated expressions for the coproduct.

The representation of the universal T-matrix in terms of the exponential factors, given in [16] is one of the forms of the Gauss factorization also. Connection of the Gauss decomposition with Borel one, which is fundamental for the Borel-Weil construction [30] in the representation theory also has the quantum group analogs [18, 27-29].

In the dual case of quantum algebras the Gauss decomposition for the L-matrices $L^{\pm}$, which was used in [4] as the very defining relations, proves to be useful in the consideration [17] of the equivalence of the R-matrix formulation of the quantum affine algebra $U_q(\widehat{sl}_2)$ [31] and the so called "new realization" [32]. The structure and some properties of the R-matrix may be conveniently described by the Gauss decomposition [24] also. Moreover the Gauss factorization was successfully used in the construction of the quantum group valued coherent states [18] for the compact quantum algebras.

The simpler structure of the commutation rules among the Gauss decomposition generators of the quantum groups simplifies [20, 21] the problem of their q-bosonization [34, 36] (that is their realization by the creation and annihilation operators of the quantum deformed oscillator [37-39]). For the dual objects – quantum algebras (or quantum deformation of universal enveloping algebras of the classical Lie algebras) this problem was considered e.g. in [40,41] and for q-superalgebras in [42,43]. The q-bosonization of the L-operators for the quantum affine algebra $U_q(\widehat{sl}_2)$, related to the Gauss decomposition, was analyzed in [22], where connection of the "new realization" of this algebra at the level 1 with the free field realization of the Zamolodchikov-Faddeev algebra formulated in the framework of the QISM was established.

To our surprise many relations among the entries of the quantum matrix T in terms of the Gauss generators become the elegant analogs of the standard classical results [33]. The distinctions appeared in these q-analogs consist essentially in the appearance of the additional q-factors and/or $q-$names only, in the replacement of the determinants and minors on the quantum determinants and q-minors, respectively and so on. This kind of results were found by many authors and rediscovered numerously (see, for example, [6,9-11,19-21,27-28]).

In this work we restrict our consideration to those properties of the Gauss factorization of the quantum group generator matrix T corresponding to the classical series of the Lie groups and supergroups which can be obtained by the elementary linear algebra methods and the R-matrix formalism using its fundamental representation. This allows us to avoid many difficult and/or cumbersome problems, connected, for example with the invertibility of some generators and $q-$minors, with the questions of the original Hopf algebra structure equivalence with those generated by new generators appeared in the Gauss decomposition [13,23], or the questions about connection of the new generators of the quantum group with the entries of the $L^{\pm}$-matrices in terms of which dual quantum algebras are described in the R-matrix approach [4]. We leave aside the higher (co)representation Gauss decomposition and the block matrix cases (see, however Sec.5 on the supergroups).

The paper is set up as follows. The Gauss decomposition of the $GL_q(n)$ is given in Sec.2. In the next Sec.3 we consider the quantum orthogonal and symplectic groups for their defining relations includes an additional quadratic constraint . An example of $Sp_q(2)$ is written in Sec.4 in detail. Some peculiarities of the quantum supergroups (Z_2-graded tensor product and s-determinant) are mentioned in Sec.5.

2. Gauss factorization for the quantum group $GL_q(n)$. It was mentioned above that the matrix FRT-relation (1) encodes quadratic (commutation) relations between quantum group generators. The R-matrix for the quantum group $GL_q(n)$ (A_l series) is the lower triangular number $n^2 \times n^2$-matrix [3,4]:

$$R = \sum_{i\neq j}^{n} e_{ii} \otimes e_{jj} + q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \lambda \sum_{i<j}^{n} e_{ji} \otimes e_{ij}, \qquad (2)$$

where $\lambda = q - q^{-1}, q \neq 1$, and e_{kl} are the standard matrix units that is the $(n \times n)$-matrices all entries of which are zeros except the element $(e_{kl})_{kl} = 1$. It is convenient to parametrize the rows and columns of the R-matrix, acting in the space $\mathbf{C}^n \otimes \mathbf{C}^n$, by the pair of indices. With a such parameterization the element of the R-matrix (2), which stays on the intersection of

the row (m,n) and the column (p,r), has the form

$$R_{mn,pr} = (1+(q-1)\delta_{mn})\delta_{mp}\delta_{nr} + \lambda\theta(m-n)\delta_{np}\delta_{mr}, \tag{3}$$

where

$$\theta(m-n) = \begin{cases} 1, & \text{if} \quad m > n \\ 0, & \text{if} \quad m \leq n \end{cases},$$

and the FRT-relation can be represented in the elementwise form as

$$\sum_{j,k=1}^{n} R_{mi,jk} t_{jp} t_{kr} = \sum_{j,k=1}^{n} t_{ik} t_{mj} R_{jk,pr}. \tag{4}$$

Due to (3), the relations (4) may be rewritten as follows

$$\begin{aligned} t_{ij} t_{ik} &= q t_{ik} t_{ij}, \qquad t_{ik} t_{lj} = t_{lj} t_{ik}, \\ t_{ik} t_{lk} &= q t_{lk} t_{ik}, \qquad [t_{ij}, t_{lk}] = \lambda t_{ik} t_{lj}, \end{aligned} \tag{5}$$

where $1 \leq j < k \leq n,\ 1 \leq i < l \leq n$. It follows from these relations that the algebra F_q has a rich subalgebra structure. In particular, if we omit in the q-matrix $T = (t_{mn})$ k arbitrary rows and k arbitrary columns $(k < n)$ then we get the q-matrix related to the $GL_q(n-k)$ subalgebra. In other words the quadratic commutation relations, defining $GL_q(n)$ are uniquely fixed by the condition that four generators which stays at the corners of each rectangular "drawn" in T are subject to the simple well known relations of the quantum group $GL_q(2)$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad \begin{aligned} ab &= qba, \quad ac = qca, \qquad bc = cb, \\ bd &= qdb, \quad cd = qdc, \quad [a,d] = \lambda bc. \end{aligned} \tag{6}$$

The **quantum determinant** is given by the expression [4]

$$D_q(T) = \det_q T = \sum_{\sigma} (-q)^{l(\sigma)} t_{1\sigma(1)} t_{2\sigma(2)} \cdot \ldots \cdot t_{n\sigma(n)}, \tag{7}$$

where $l(\sigma)$ is the length of the substitution $\sigma \in \mathbf{S}_n$. The q-determinant $\det_q T$ is a central element for the quantum group $GL_q(n)$ [4,6]. For the quantum group $GL_q(2)$ the quantum determinant is equal to

$$\det_q T = ad - qbc = da - q^{-1} bc.$$

The invertibility of the quantum determinant $\det_{\mathrm{q}} T$ is the necessary condition for introducing the Hopf algebra structure into the algebra $GL_q(n)$. In the following we suppose that this condition is fulfilled.

The **Gauss decomposition** is the quantum matrix factorization of the following form

$$T = T_L T_D T_U = T_L T^{(+)} = T^{(-)} T_U, \tag{8}$$

where $T_L = (l_{ik})$ is the strictly lower triangular matrix (with unities at the main diagonal $l_{kk} = 1$); $T_D = \mathrm{diag}(A_{kk})$; $T_U = (u_{ik})$ is the strictly upper triangular matrix ($u_{kk} = 1$); $T^{(+)} := T_D T_U$ and $T^{(-)} = T_L T_D$. It is an easy exercise to get for the $GL_q(2)$ (6) $l_{21} = c/a, u_{12} = b/qa, A_{11} = a, A_{22} = det_q T/a$.

As in the classical case, with the mutually commuting entries of the matrix T, the Gauss algorithm to construct the triangular factorization (see for example [33]) can be applied for the quantum matrix as well. To realize the Gauss algorithm let us introduce strictly lower triangular $n \times n$-matrix W_L and the upper triangular matrix $T^{(+)}$ by the equation:

$$W_L T = T^{(+)}. \tag{9}$$

Let $w_k = (w_{k,1}, w_{k,2}, ..., w_{k,k-1})$ and $t_k = (t_{k,1}, t_{k,2}, ..., t_{k,k-1})$ be the parts of the k-th rows of the matrix W_L and T, respectively, and $T_{(k)}$ denotes the matrix which was received from T by omitting the last $(n-k)$ rows and $(n-k)$ columns. Then in view of the upper triangularity of the matrix $T^{(+)}$ the condition (9) gives the system of the linear equations

$$w_k T_{(k-1)} = -t_k \tag{10}$$

on elements of the row w_k with generators of the algebra F_q as coefficients. If the q-matrix $T_{(k)}$ is invertible (that is $\mathrm{det_q} T_{(k)} \neq 0$), then the solution of this system has the form

$$w_k = -t_k T^{-1}_{(k-1)}.$$

We note that the needed for the existence of a such solution condition corresponds completely to the classical condition that all diagonal minors of the classical matrix are different from the zero. The elements of the inverse matrix are given by the formula [4,6]:

$$(T^{-1})_{ij} = (-q)^{i-j} (\det_q T)^{-1} M_q(i,j), \tag{11}$$

where $M_q(j,i)$ is the q-minor, that is the quantum determinant of the matrix which is obtained from T by omitting of the i-th row and j-th column. Thus all the elements of the matrix W_L are uniquely defined. The condition (9) allows to find all the nonzero elements

of the matrix $T^{(+)}$. In particular the diagonal elements except $(T^{(+)})_{11} = t_{11}$, have the form

$$\begin{gathered}
(T^{(+)})_{ii} = \sum_{k=1}^{i} (W_L)_{ik} t_{ki} = -\sum_{l,k=1}^{i-1} (-q)^{l-k} t_{il} (\det_q T_{(i-1)})^{-1} M_q(l,k) t_{ki} + t_{ii} = \\
(\det_q T_{(i-1)})^{-1} \left[\sum_{l,k=1}^{i-1} (-q)^{l-k+1} t_{il} M_q(l,k) t_{ki} + (\det_q T_{(i-1)}) t_{ii} \right] = \\
(\det_q T_{(i-1)})^{-1} \left[\sum_{k=1}^{i} (-q)^{i-k} M_q(k,i) t_{ki} \right] = (\det_q T_{(i-1)})^{-1} (\det_q T_{(i)}).
\end{gathered} \tag{12}$$

To derive this relation we take into account the relations

$$M_q(n,n) t_{nk} = q t_{nk} M_q(n,n), \quad k < n,$$

and the decomposition $(1 \leq k \leq n)$

$$\det_q T = \sum_{j=1}^{n} (-q)^{j-k} t_{kj} M_q(k,j) = \sum_{j=1}^{n} (-q)^{k-j} M_q(k,j) t_{jk}.$$

These and some other useful formulas are given in [6]. Let us point out that the relation

$$(T^{(+)})_{ii} = (\det_q T_{(i-1)})^{-1} (\det_q T_{(i)}) \tag{13}$$

is the direct q-analog of the classical formulas [33]. Then it is easy to see that the diagonal elements $(T^{(+)})_{ii}$ of the matrix $T^{(+)}$ are mutually commuting. Using the commutation relations between the elements t_{ij} of the matrix T and their minors (which as a rule proved in [6, 9]) one can find the commutation rules between elements $(T^{(+)})_{ij}$ of the matrix $T^{(+)}$. The matrix W_L being strictly lower triangular can be inverted. The elements $(T_L)_{ij}$ of the inverse matrix $(T_L) = (W_L)^{-1}$ are polynomials on the elements of the matrix W_L. From the relation (9) the desired decomposition of the q-matrix T follows

$$T = T_L T^{(+)}. \tag{14}$$

In the same way one can find the matrix W_U, such that $T^{(-)} = T W_U$, where $T^{(-)}$ is the lower triangular matrix. The decomposition of the matrix T takes now the form $(W_U^{-1} = T_U)$

$$T = T^{(-)} T_U. \tag{15}$$

If we multiply from the right the relation (14) by the matrix W_U, $T^{(-)} = T W_U$, then

$$T^{(-)} = T W_U = T_L T^{(+)} W_U = T_L (T^{(+)} T_U^{-1}) = T_L T_D,$$

where $T_D = T^{(+)}T_U^{-1}$ is the diagonal matrix with the elements given by (13). One has also $T^{(+)} = T_D T_U$. Hence, we get the Gauss factorization (8) of the q-matrix $T = T_L T_D T_U$.

The considered above factorization procedure allows to obtain the expressions for the elements of all matrices participated in the decomposition in terms of the quantum group $GL_q(n)$ generators and principal minors. Unfortunately such factorization procedure is rather cumbersome for it require to find the commutation rules between the elements of the received matrices. For the quantum group $GL_q(n)$ most of these formulas can be extracted from [6]. At the same time for quantum groups corresponding to other classical series explicit formulas for quantum determinants and minors as well as the commutation rules are much less known. This difficulty can be avoided using the contraction procedure which for the quantum group case was considered, for example, in [44, 45, 47].

In the fundamental representation the Cartan elements h_i of the Lie algebra $gl(n)$ are represented by the $n \times n$ matrices $h_i = \frac{1}{2n} e_{ii}$. In the process of the quantum deformation they are not changed. Moreover they retain to be elements of the Lie type (primitive) that is their coproduct has the form $\Delta(h_i) \equiv H_i = h_i \otimes 1 + 1 \otimes h_i$. Hence, on the Cartan elements the coproduct is coincide with the opposite one $\Delta(h_i) = \Delta'(h_i)$, where $\Delta' = P \circ \Delta$ and we have

$$[R, H_i] = 0. \tag{16}$$

Let us apply to the FRT-relation (1) the similarity transformation with the matrix

$$K_\gamma = \exp(\sum_{i=1}^{n} \gamma_i h_i) \otimes \exp(\sum_{i=1}^{n} \gamma_i h_i) = \exp(\sum_{i=1}^{n} \gamma_i \Delta(h_i)) = \exp(\sum_{i=1}^{n} \gamma_i H_i),$$

where the number coefficients γ_i are strictly ordered $\gamma_1 > \gamma_2 > \ldots > \gamma_n > 0$. In view of the relation (16), such similarity transformation affects only the matrices $T_l, (l = 1, 2)$:

$$T_l \longrightarrow K_\gamma T_l K_\gamma^{-1} = \mathrm{diag}(e^{\gamma_i}) T \mathrm{diag}(e^{-\gamma_i}), \qquad t_{ij} \longmapsto t_{ij} e^{\gamma_i - \gamma_j}.$$

Let us introduce two sets of the new generators

$$t_{ij}^{(+)} = \begin{cases} t_{ij} e^{\gamma_i - \gamma_j} & i \le j \\ t_{ij} & i > j \end{cases} ; \quad t_{ij}^{(-)} = \begin{cases} t_{ij} & i < j \\ t_{ij} e^{\gamma_i - \gamma_j} & i \ge j \end{cases} .$$

Let $\gamma_i - \gamma_j = \gamma_{ij}\varepsilon$, where $\gamma_{ij} > 0$, if $i < j$ and $\gamma_{ij} < 0$, if $i > j$. When $\varepsilon \to \infty$ ($\varepsilon \to -\infty$) in the set $\{t_{ij}^{(+)}\}$ $\{(t_{ij}^{(-)})\}$ all the matrix elements with $i > j$ ($i < j$) are disappeared. Thus we constructed the homomorphisms of the algebra F_q into the algebras $F_q^{(\pm)}$, generated by

the upper and lower triangular q-matrices. In this case the commutation rules for the new generators are defined by the same R-matrix:

$$R\widetilde{T}_1^{(\pm)}\widetilde{T}_2^{(\pm)} = \widetilde{T}_2^{(\pm)}\widetilde{T}_1^{(\pm)}R. \tag{17}$$

Similar contraction procedure allows to find the homomorphisms $\widetilde{T}^{(\pm)} \to \widetilde{A}$ of the quantum groups generated by the entries of matrices $\widetilde{T}^{(\pm)}$ into the group generators of which are the elements of the diagonal matrix T_D. The commutation rules between elements of this group are also determined by the FRT-relation

$$R\widetilde{A}_1\widetilde{A}_2 = \widetilde{A}_2\widetilde{A}_1R, \qquad (A_{ij} = \delta_{ij}A_{ij}), \tag{18}$$

which due to of the structure of the R-matrix for the $GL_q(n)$ is equivalent to the relation

$$\widetilde{A}_1\widetilde{A}_2 = \widetilde{A}_2\widetilde{A}_1. \tag{19}$$

This means the commutativity of the diagonal matrix elements and can be easily proved using also the FRT-relation (4) and the R-matrix elements (3).

Let us return to the relation (17) and consider once more contraction with the matrices $K_{1\gamma} = \exp(\sum_{i=1}^n \gamma_i h_i) \otimes 1$ (or $K_{2\gamma} = 1 \otimes \exp(\sum_{i=1}^n \gamma_i h_i)$). The similarity transformation with these operators leaves without changes in the equality (17) the matrix $\widetilde{T}_2^{(\pm)}$ ($\widetilde{T}_1^{(\pm)}$), but transforms the R-matrix $R \to R_\gamma$. For example, the contraction which maps the matrix $T^{(-)}$ into the diagonal ones ($\varepsilon \to \infty$ under the mentioned above type of ordering in the set $\{\gamma_i\}$), the matrix R goes into diagonal matrix $R_\gamma \to R_D$ (where R_D is the diagonal part of the R-matrix). Finally we get the relation

$$R_D\widetilde{A}_1\widetilde{T}_2^{(-)} = \widetilde{T}_2^{(-)}\widetilde{A}_1R_D. \tag{20}$$

In a similar way one gets the R-matrix relation for the matrix $\widetilde{T}^{(+)}$

$$R_D\widetilde{T}_1^{(+)}\widetilde{A}_2 = \widetilde{A}_2\widetilde{T}_1^{(+)}R_D. \tag{21}$$

Using the transposition operator $\mathcal{P}$ $\quad(\mathcal{P}_{ij,kl} = \delta_{il}\delta_{jk})$, such that

$$\mathcal{P}^2 = \mathcal{P},\ \mathcal{P}T_1\mathcal{P} = T_2, \quad \mathcal{P}T_2\mathcal{P} = T_1, \quad \mathcal{P}R_D\mathcal{P} = R_D, \tag{22}$$

we receive the relations (20-21) with the replacement $1 \leftrightarrow 2$.. Finally, after the similarity transformation of the FRT-relation (1) with the operators $K_{\gamma 1}, K_{\gamma 2}$, we get the relation

$$R_D\widetilde{T}_1^{(+)}\widetilde{T}_2^{(-)} = \widetilde{T}_2^{(-)}\widetilde{T}_1^{(+)}R_D. \tag{23}$$

It can be shown that commutation rules of the new generators defined uniquely from the commutation relations of the initial generators. This means that the quantum groups, obtained by this contraction procedure, are isomorphic to the quantum groups, generated by the respective factors of the Gauss decomposition (8) for the quantum group $GL_q(n)$: $\widetilde{T}^{(+)} \simeq T^{(+)}, \widetilde{A} \simeq T_D, \widetilde{T}^{(-)} \simeq T^{(-)}$. Thus we see that commutation rules between generators of the quantum groups T_U and T_L can be obtained from the relations (17-23). For example, from the relation (17) for $T^{(+)}$ and the decomposition $T^{(+)} = T_D T_U$ we have

$$RT_1^{(+)}(T_D)_2 T_{U2} = T_2^{(+)}(T_D)_1 T_{U1} R$$

or, with the help of the commutation relations (21),

$$RR_D^{-1} T_{D2} T_1^{(+)} R_D T_{U2} = R_D^{-1} T_{D1} T_1^{(+)} R_D T_{U1} R.$$

From the relation (18), and taking into account that $[R_D, R] = 0$, we have finally

$$RT_{U1} R_D T_{U2} = T_{U2} R_D T_{U1} R, \tag{24}$$

which is the variant of the reflection equations considered for example in [47-50]. Analogous equation

$$RT_{L1} R_D^{-1} T_{L2} = T_{L2} R_D^{-1} T_{L1} R \tag{25}$$

can be obtained for the matrix T_L. Because the diagonal matrices R_D and T_{Di} ($i = 1, 2$) commutes with each other, from the relations (20) and (21) follows the equality

$$R_D T_{D1} T_{L2} = T_{L2} T_{D1} R_D \tag{26}$$

$$R_D T_{U1} T_{D2} = T_{D2} T_{U1} R_D \tag{27}$$

(and similar relation with substitution $1 \leftrightarrow 2$). If we substitute $T^{(\pm)}$ into the relation (23) and take into account relations (26) and (27) we received

$$T_{U1} T_{L2} = T_{L2} T_{U1}. \tag{28}$$

Thus all elements of the q-matrix T_U commutes with every element of the q-matrix T_L.

On the other hand if the relations (17-27) are valid then every decomposition (8) fulfills the FRT-relation. For example, for the matrix decomposition $T = T^{(-)} T_U$ we have

$$\begin{gathered} RT_1 T_2 - T_2 T_1 R = RT_1^{(-)}(T_{U1} T_2^{(-)} - T_2^{(-)} R_D^{-1} T_{U1} R_D) T_{U2} - \\ T_2^{(-)}(T_{U2} T_1^{(-)} - T_1^{(-)} R_D^{-1} T_{U2} R_D) T_{U1} R + \\ RT_1^{(-)} T_2^{(-)} R_D^{-1}(T_{U1} R_D T_{U2} - R^{-1} T_{U2} R_D T_{U1} R), \end{gathered}$$

Three expressions in the brackets () are zeros due to (27) and (24).

Thus the relations (19),(24)-(28) give complete list of the commutation relations for the elements of the q-matrices T_L, T_D and T_U from the Gauss decomposition of the original matrix T for the quantum group $GL_q(n)$. The relations (17), (24)-(25), and easily obtained from (26) and (27) relations

$$R_D T_1^{(+)} T_{L2} = T_{L2} R_D T_1^{(+)}; \tag{29}$$

$$T_2^{(-)} R_D{}^{-1} T_{L1} = T_{L1} T_2^{(-)} R_D{}^{-1}, \tag{30}$$

give the full set of commutation rules between elements of the matrices $\{T_L, T^{(+)}\}$ and $\{T^{(-)}, T_U\}$. This allows to consider each of this three sets of elements as the new basis of generators for the quantum group $GL_q(n)$. The basis $\{T_L, T_D, T_U\}$ is particularly convenient one. Diagonal elements of the matrix T_D remind the Cartan generators of the simple Lie algebra. Quantum determinant (7) is a central element of the quantum group $GL_q(n)$ and, as in the case of number matrices can be expressed as the product of diagonal elements

$$\det_q T = \prod_{i=1}^{n} (T_D)_{ii}. \tag{31}$$

Of course $\det_q T$ commutes with all new generators, this can be easily proved for the RHS using the relations (26),(27). For example, if we commute the element u_{mp} in succession with every factor in $\prod_{i=1}^{n}(T_D)_{ii}$ we get due to the relations (26),(27) the following equalities

$$u_{mp} \prod_{i=1}^{n} (T_D)_{ii} = \prod_{i=1}^{n} (T_D)_{ii} \frac{R_{pi,pi}}{R_{mi,mi}} u_{mp} = \prod_{i=1}^{n} (T_D)_{ii} u_{mp},$$

because $\prod_{i=1}^{n} R_{pi,pi} = q$ for every $1 \leq p \leq n$. Commutation relations of the T_D diagonal elements are defined by the diagonal blocks of the R-matrix (3) (cf (26), (27)). The product of these blocks is proportional to the unit matrix, hence the product of the T_D diagonal elements is central (which is not the case of the multiparametric R-matrix).

3. Orthogonal and symplectic quantum groups. In this section orthogonal and symplectic quantum groups (that is the groups of the series B_l, C_l, D_l) are considered and we shell omit often their names in what follows. The commutation relations between generators of these groups are determined by the FRT-relation with the R-matrix [4,5]

$$\begin{gathered} R = q \sum_{\substack{i=1 \\ i \neq i'}}^{N} e_{ii} \otimes e_{ii} + e_{\frac{N+1}{2},\frac{N+1}{2}} \otimes e_{\frac{N+1}{2},\frac{N+1}{2}} + \sum_{\substack{i,j=1 \\ i \neq j,j'}}^{N} e_{ii} \otimes e_{jj} \\ + q^{-1} \sum_{\substack{i=1 \\ i \neq i'}}^{N} e_{i'i'} \otimes e_{ii} + \lambda \sum_{\substack{i,j=1 \\ i>j}}^{N} e_{ij} \otimes e_{ji} - \lambda \sum_{\substack{i,j=1 \\ i>j}}^{N} q^{\rho_i - \rho_j} \varepsilon_i \varepsilon_j e_{ij} \otimes e_{i'j'}. \end{gathered} \tag{32}$$

in which the second term takes place in the B – series only. One has in (32) $N = 2n$ for the groups of C - and D - series, $N = 2n+1$ for the B – series; $\epsilon_i = 1$ for B – and D – series, $\varepsilon_i = \begin{cases} 1, & i \leq N/2, \\ -1, & i > N/2. \end{cases}$ for C – series; $i' = N+1-i$; $\{\rho_i\}$ is a set of numbers fixed for each group [4, 5].

In the quantum group definition one introduces the additional condition [4]

$$TCT^tC^{-1} = 1 = CT^tC^{-1}T, \tag{33}$$

which is a quantum analogs of the known conditions for matrices of the orthogonal and symplectic Lie groups in the fundamental representation. In (33) T is the quantum group q-matrix; T^t is the transposed matrix, C is the number matrix [4, 5]

$$C = C_0 q^{\rho}, \quad \rho = \mathrm{diag}(\rho_1, \rho_2, ..., \rho_N), \quad (C_0)_{ij} = \epsilon_i \delta_{ij'}.$$

The relation (33) has more transparent form in terms of the group generators

$$\sum_{k=1}^{N} \varepsilon_k q^{\rho_{k'}} t_{ik} t_{jk'} = \varepsilon_{j'} \delta_{ij'} q^{\rho_j}, \tag{34}$$

$$\sum_{l=1}^{N} \varepsilon_l q^{\rho_{l'}} t_{li} t_{l'j} = \varepsilon \varepsilon_{i'} \delta_{i'j} q^{-\rho_i}. \tag{35}$$

where ρ_i and ε_i were defined above, $\varepsilon = 1$ for B - and D - series group, $\varepsilon = -1$ for C - series. Among the additional conditions (34),(35) linear independent from the commutation relations defined by the FRT-equation (1) are only those in which right hand side is unequal to zero (in other words, "diagonal" part of the relation (33)).

As for the $GL_q(n)$ case, the function algebras F_q of other series quantum groups also have rich subalgebra structures. For example, generators which stand in the q-matrix T on crosses of rows with indices $(i_1, i_2, .., i_k)$ and columns with indices $(j_1, j_2, .., j_k)$ form $GL_q(k)$ subalgebra in F_q if $k \leq n$ and among row numbers as well as among column numbers there are no pairs (i_m, i_l) with $i_m = (i_l)' = N+1-i_l$).

The Gauss factorization of the orthogonal and symplectic groups can be carried out by the same method as in Sec.2 for the $GL_q(n)$. The commutation relations of the generators are determined by the same formulae (19)-(21), (24)-(28) in the basis (T_L, T_D, T_U) and by (17), (24)-(25) in the basis $(T_L, T^{(+)})$ and $(T^{(-)}, T_U)$. Particular properties of orthogonal and symplectic R-matrices mean, that in deriving the relation (19) from (18), it is necessary also to take into account that in the considered case

$$(T_D)_{ii}(T_D)_{i'i'} = (T_D)_{jj}(T_D)_{j'j'} \quad 1 \leq i, j \leq n; \quad i' := N+1-i. \tag{36}$$

This relation is not a restriction for the additional conditions (33)require stronger relation

$$(T_D)_{ii}(T_D)_{i'i'} = 1, \quad 1 \le i \le N. \tag{37}$$

Using the commutation relations one can prove that the additional conditions for the matrices $T^{(\pm)}$ have the same form (33) as for T, or in terms of the matrix elements (34),(35). Substituting, for example, $T^{(-)} = T_L T_D$ in these expressions one can obtain the additional conditions for the generators $(T_L)_{ij}$:

$$\sum_{k=1}^{N} \varepsilon_k q^{\rho_{k'}} \frac{R_{kk';kk'}}{R_{kn;kn}} (T_L)_{mk}(T_L)_{nk'} = \varepsilon_{n'} \delta_{mn'} q^{\rho_n}; \tag{38}$$

$$\sum_{l=1}^{N} \varepsilon_l q^{\rho_{l'}} \frac{R_{mn;mn}}{R_{ml';ml'}} (T_L)_{lm}(T_L)_{l'n} = \varepsilon \varepsilon_{m'} \delta_{mn'} q^{-\rho_m}. \tag{39}$$

where $(T_L)_{ii} = 1;\ (T_L)_{ij} = 0$, at $i < j$. The same relations are valid for T_U - matrix elements. It is essential that the sums on the left hand sides of (38),(39) include linear by generators terms. It allows us to exclude dependent generators from a generator list. Finally the number of generators is equal to the dimension of the related Lie group. We shell illustrate independent generator choice in the next section considering the quantum group $Sp_q(2)$ as an example.

4. An example: symplectic quantum group $Sp_q(2)$. In this Sec. some details of the Gauss factorization of symplectic and orthogonal quantum groups are considered using $Sp_q(2)$ as an example. One can take also even simpler case of the quantum group $SO_q(3)$ with the matrix $(C)_{ij} = (-q)^{i-2}\delta_{ij'}, i,j = 1,2,3$ in (33). However, due to the fusion procedure [53] it corresponds to the spin 1 corepresentation of the $SU_q(2)$ quantum group (or $GL_q(2)$ if $det_q T$ is not 1). Hence the Gauss decomposition follows from the one of (6) by tensoring and projecting with the diagonal part

$$(A_{11}, A_{22}, A_{33}) = (a^2, \quad det_q T, \quad (det_q T)^2/a^2),$$

while the additional condition (33) is a consequence of $T^t \epsilon_q T = \epsilon_q det_q T$ for (6) [61] . The entries of T_L and T_U are expressed in terms of two generators u_{12} and l_{21} ($[2]_q = q + 1/q$)

$$l_{31} = l_{21}^2/[2]_q = c^2/a^2; \qquad l_{32} = l_{21}/q;$$
$$u_{13} = u_{12}^2/[2]_q = b^2/q^4a^2; \qquad u_{23} = qu_{12}.$$

The R - matrix for the $Sp_q(2)$ group is

$$R = \begin{pmatrix}
q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & q^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \lambda & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\lambda/q & \cdot & \cdot & q^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \lambda/q^3 & \cdot & \cdot & \lambda X & \cdot & \cdot & q^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \lambda Y & \cdot & \cdot & \lambda/q^3 & \cdot & \cdot & -\lambda/q & \cdot & \cdot & q^{-1} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q
\end{pmatrix} \qquad (40)$$

where $X = (1 + q^{-2});\ Y = (1 + q^{-4})$. This matrix has more nonzero elements than the same rank quantum group $GL_q(4)$ R-matrix, in which all the diagonal terms with q^{-1} are changed to 1, and all the terms with multiplication by the negative degree of parameter q out of the diagonal are zero. The complicated structure of the R-matrix causes complicated form of the commutation relations. The last remark refers only to special commutation relations between elements whose row or/and column indices are connected by the relations $i = j'$, where $j' = N + 1 - j$. In the $Sp_q(2)$ case, for which $N = 2n = 4$, $j' = 5 - j$. Other generators have the $GL_q(2)$ commutation relations as they are elements of the $GL_q(2)$-subalgebras of

the quantum group $Sp_q(2)$. We write down explicitly the "special" relations only:

$$\begin{gathered}
t_{i1}t_{i4}=q^2t_{i4}t_{i1},\quad t_{i2}t_{i3}=q^2t_{i3}t_{i2}+\lambda t_{i1}t_{i4},\quad 1\le i\le 4;\\
t_{1i}t_{4i}=q^2t_{4i}t_{1i},\quad t_{2i}t_{3i}=q^2t_{3i}t_{2i}+\lambda t_{1i}t_{4i},\quad 1\le i\le 4;\\
q^{-1}t_{im}t_{i'n}=t_{i'n}t_{im}+\lambda q^{\rho_i}\epsilon_i\sum_{j<i}q^{-\rho_j}\epsilon_j t_{jm}t_{j'n},\quad m\ne n';\ \begin{cases} i=1,2;\quad n<m;\quad \text{or}\\ i=3,4;\quad n>m;\end{cases};\\
q^{-1}t_{im}t_{i'n}-\lambda t_{i'm}t_{in}=t_{i'n}t_{im}+\lambda q^{\rho_i}\epsilon_i\sum_{j<i}q^{-\rho_j}\epsilon_j t_{jm}t_{j'n},\\
m\ne n',\ i=1,2;\ n>m;\\
t_{im}t_{jm'}=q^{-1}t_{jm'}t_{im}-\lambda q^{-\rho_m}\epsilon_m\sum_{k>m}q^{\rho_k}\epsilon_k t_{jk'}t_{ik},\quad i\ne j';\ \begin{cases} m=1,2;\quad i>j;\quad \text{or}\\ m=3,4;\quad i<j;\end{cases};\\
t_{im}t_{jm'}-\lambda t_{jm}t_{im'}=q^{-1}t_{jm'}t_{im}-\lambda q^{-\rho_m}\epsilon_m\sum_{k>m}q^{\rho_k}\epsilon_k t_{jk'}t_{ik},\\
i\ne j',\ m=1,2;\ i<j;\\
q^{-1}t_{ij}t_{i'j'}-\lambda q^{\rho_i}\epsilon_i\sum_{k<i}q^{-\rho_k}\epsilon_k t_{kj}t_{k'j'}=\\
=q^{-1}t_{i'j'}t_{ij}-\lambda q^{-\rho_j}\epsilon_j\sum_{k>j}q^{\rho_k}\epsilon_k t_{i'k'}t_{ik},\ \begin{cases} j=1,2;\quad i=3,4;\quad \text{or}\\ j=3,4;\quad i=1,2;\end{cases};\\
q^{-1}t_{ij}t_{i'j'}-\lambda t_{i'j}t_{ij'}-\lambda q^{\rho_i}\epsilon_i\sum_{k<i}q^{-\rho_k}\epsilon_k t_{kj}t_{k'j'}=\\
=q^{-1}t_{i'j'}t_{ij}-\lambda q^{-\rho_j}\epsilon_j\sum_{k>j}q^{\rho_k}\epsilon_k t_{i'k'}t_{ik},\ \ i,j=1,2;
\end{gathered}\tag{41}$$

In these formulas

$$\epsilon_1=\epsilon_2=1,\ \epsilon_3=\epsilon_4=-1,\quad \rho_i=(\rho_1,\rho_2,\rho_3,\rho_4)=(2,1,-1,-2).$$

Adding the additional relations (34),(35) we get the complete set of commutation relations for the $Sp_q(2)$ quantum group (the FRT-relation together with the additional constraint (33) yield an overcomplete list).

As in Sec.2 for realization of the classical Gauss algorithm we must find matrices W_L and W_U solving the system of equations (10). E. g. for W_L one has:

$$\begin{gathered}
w^L_{21}=-t_{21}D_q^{-1}[{}^1_1];\quad w^L_{31}=(D_q[{}^{2,3}_{1,3}]-\lambda D_q[{}^{1,4}_{1,2}])D_q^{-1}[{}^{1,2}_{1,2}];\quad w^L_{32}=-D_q[{}^{1,3}_{1,2}]D_q^{-1}[{}^{1,2}_{1,2}];\\
w^L_{41}=-q^2t_{41}D_q^{-1}[{}^1_1];\quad w^L_{42}=-q^2t_{31}D_q^{-1}[{}^1_1];\quad w^L_{43}=t_{21}D_q^{-1}[{}^1_1];
\end{gathered}\tag{42}$$

In these expressions the q-minors $D_q[{}^{i_1,i_2,...,i_k}_{j_1,j_2,...,j_k}]$ are calculated by the GL_q-rules (7) for a matrix obtained from T by removing all the rows and columns except those with indices $(i_1,...,i_k)$ $(j_1,...,j_k)$ respectively. In particular,

$$D_q[{}^1_1]=t_{11};\qquad D_q[{}^{1,2}_{1,2}]=t_{11}t_{22}-qt_{12}t_{21}.$$

Using (9) and the additional conditions (34),(35) for the matrix $T^{(+)}$ one obtains

$$T^{(+)} = \begin{pmatrix} D_q[{}^1_1] & t_{12} & t_{13} & t_{14} \\ 0 & D_q^{-1}[{}^1_1]D_q[{}^{1,2}_{1,2}] & D_q^{-1}[{}^1_1]D_q[{}^{1,2}_{1,3}] & D_q^{-1}[{}^1_1](t_{11}t_{24} - q^2 t_{14}t_{21}) \\ 0 & 0 & D_q^{-1}[{}^{1,2}_{1,2}]D_q[{}^1_1] & -D_q^{-1}[{}^1_1]t_{12} \\ 0 & 0 & 0 & D_q^{-1}[{}^1_1] \end{pmatrix}$$

As in the classical commutative case, the diagonal elements for quantum groups (see the $GL_q(n)$-case above) are the ratios of diagonal minors

$$D_q^{-1}[{}^{1,2,\ldots,k-1}_{1,2,\ldots,k-1}]D_q[{}^{1,2,\ldots,k}_{1,2,\ldots,k}].$$

In the presented matrix these ratios are simplified using particular properties of the symplectic quantum groups. Namely, calculating the $T^{(+)}$ matrix elements on the places related to the diagonal GL_q - minors the expressions of the following form appear

$$D_q^{sp}[{}^{1,2,\ldots,k}_{1,2,\ldots,k}] = \sum_\sigma (-q)^{l(\sigma)} q^{l'(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdot \ldots \cdot t_{k,\sigma(k)}, \tag{43}$$

which have an additional factor $q^{l'(\sigma)}$ in comparison with (7). Here $l'(\sigma)$ is the number of transpositions of "specific" elements (transposition index). For example, $l'(1,3,2,4) = 1$ as 2 and $3 = 2'$ are transposed, $l'(1,2,4,3) = 0$. The expressions (43) for $i = 3,4$ can be simplified in the following manner

$$D_q^{sp}[{}^{1,2,3}_{1,2,3}] = D_q^{sp}[{}^1_1] \qquad D_q^{sp}(T) = D_q^{sp}[{}^{1,2,3,4}_{1,2,3,4}] = 1. \tag{44}$$

So the formula (43) can be naturally chosen as a quantum determinant definition in the symplectic case. This definition is in agreement with the one given by a geometric consideration [11].

Non zero elements of the matrix $W_L^{-1} = T_L$ have the form (we remind that $T_L = (l_{ij}), T_D = (A_{ii}), T_U = (u_{ij})$)

$$\begin{gathered} (T_L)_{21} = -w_{21}; \quad (T_L)_{32} = -w_{32}; \quad (T_L)_{43} = -w_{43}; \\ (T_L)_{31} = w_{32}w_{21} - w_{31} = t_{31}D_q^{-1}[{}^1_1]; \\ (T_L)_{41} = -w_{43}w_{32}w_{21} + w_{43}w_{31} + w_{42}w_{21} - w_{41} = t_{41}D_q^{-1}[{}^1_1]; \\ (T_L)_{42} = w_{43}w_{32} - w_{42} = q^{-1}D_q[{}^{1,4}_{1,2}]D_q^{-1}[{}^{1,2}_{1,2}]; \end{gathered} \tag{45}$$

Using the definition $T^{(+)} = T_D T_U$ one can obtain the matrix elements of $T^{(+)}$. Similar procedure based on the matrix W_U leads, naturally, to the same results.

These formulas allow us to calculate commutation relations among the new basis generators induced by the Gauss decomposition. Finally, they have the form

$$
\begin{array}{ll}
(\mathrm{I}) & [A_{kk}, A_{jj}] \;=\; 0; \\[2ex]
(\mathrm{II}) & \left\{ \begin{array}{rcl} l_{21}l_{31} &=& q^2 l_{31}l_{21} + q\lambda l_{41}; \\ l_{32}l_{21} &=& q^2 l_{21}l_{32} - (q^4-1)l_{31}; \\ l_{31}l_{32} &=& q^2 l_{32}l_{31}; \\ [l_{41}, l_{ij}] &=& 0; \end{array} \right. \\[2ex]
(\mathrm{III}) & \left\{ \begin{array}{rcl} u_{12}u_{13} &=& q^2 u_{13}u_{12} + q\lambda u_{14}; \\ u_{23}u_{12} &=& q^2 u_{12}u_{23} - (q^2-q^{-2})u_{13}; \\ u_{13}u_{23} &=& q^2 u_{23}u_{13}; \\ [u_{14}, u_{kl}] &=& 0; \end{array} \right. \\[2ex]
(\mathrm{IV}) & \left\{ \begin{array}{rcl} A_m l_{ij} &=& q^{(\delta_{mj}-\delta_{mj'}-\delta_{mi}+\delta_{mi'})} l_{ij} A_m; \\ A_m u_{ij} &=& q^{(\delta_{im}-\delta_{im'}-\delta_{jm}+\delta_{jmi'})} u_{ij} A_m; \end{array} \right. \\[2ex]
(\mathrm{V}) & [u_{kl}, l_{ij}] \;=\; 0;
\end{array}
\tag{46}
$$

The additional conditions (33) cause the following connections among the new generators

$$
\begin{array}{ll}
A_{33} = A_{22}^{-1}; & A_{44} = A_{11}^{-1}; \\
l_{42} = q^2 l_{31} - l_{21}l_{32}; & l_{43} = -l_{21}; \\
u_{24} = q^2(u_{13} - u_{12}u_{23}); & u_{34} = -u_{12};
\end{array}
\tag{47}
$$

Hence, the number of independent generators decreases to 10 i.e. to the dimension of the related classical Lie group.

5. Quantum supergroups. Quantum superalgebras appeared naturally when quantum inverse scattering method [1, 52, 53] was generalized to the $\mathbf{Z}_2$- graded systems [52, 53, 54]. Related R-matrices were considered in [54, 55, 56, 57, 66] and simple examples were presented in [59]. The works [42, 43, 60] are devoted to a q-bosonization of the q-superalgebras. The simple examples of dual objects, i.e. quantum supergroups, were investigated in [60]- [67].

Necessary definitions and theorems of the "supermathematics" can be found, for instance, in [68, 69].

In this Sec. using the simple examples of the quantum supergroups: $GL_q(1|1)$ and $GL_q(2|1)$ it is shown that with minimal corrections (referring to the sign factors) most of the formulas and statements on the Gauss decomposition discussed above are survived in the supergroup case.

The FRT-relation for the quantum supergroups has the same form as (1), but matrix tensor product includes additional sign factors ($\pm$) [52, 53] related to $\mathbf{Z}_2$-grading [68, 69]. Vector $\mathbf{Z}_2$ - graded space (superspace) decomposes into the direct sum of subspaces $V_0 \oplus V_1$ of even and odd vectors on which the parity function ($p(v) = 0$ at $v \in V_0$ and $p(w) = 1$ at $w \in V_1$) is defined. As a rule, a vector basis with definite parity $p(v_i) = p(i) = 0, 1$ is using. According to this basis, the row and column parities are introduced in the matrix space End(V) [68, 69]. The tensor product of two even matrices F, G ($p(F_{ij}) = p(i) + p(j)$) has the following signs [52, 53]

$$(F \otimes G)_{ij;kl} = (-1)^{p(j)(p(i)+p(k))} F_{ik} G_{jl}. \tag{48}$$

Due to this prescription $T_2 = I \otimes T$ has the same block-diagonal form as in the standard (non super) case while $T_1 = T \otimes I$ includes the additional sign factor (-1) for odd elements standing at odd rows of blocks. The R-matrices for supergroups can be extracted, for instance, from [52], ...[59], [66]. For the $GL_q(n|m)$ quantum supergroup the R-matrix structure is the same as for the $GL_q(n+m)$ but at odd-odd rows q is changed by q^{-1}

$$R = \sum_{i,j} \left(1 - \delta_{ij}(1 - q^{1-2p(i))})\right) e_{ii} \otimes e_{jj} + \lambda \sum_{i>j} e_{ij} \otimes e_{ji}. \tag{49}$$

Let us remind that the tensor product notation in (49) refers to the graded matrices. For

convenience the R-matrices for $GL_q(1|1)$ and $GL_q(2|1)$ are presented below

$$R^{GL(1|1)} = \begin{pmatrix} q & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \lambda & 1 & \cdot \\ \cdot & \cdot & \cdot & q^{-1} \end{pmatrix}; \qquad R^{GL(2|1)} = \begin{pmatrix} q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \lambda & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^{-1} \end{pmatrix}.$$

The same contraction procedure arguments as in Sec. 2 result in homomorphisms of $T = (t_{ij})$ onto triangular $T^{(\pm)}$ and diagonal T_D matrices as well as in the related R-matrix relations. Let us present some of them pointing out peculiarities of the supergroup case as the main R-matrix properties are the same as those in Secs 2, 3.

From the relations

$$RT^{(\pm)}{}_1T^{(\pm)}{}_2 = T^{(\pm)}{}_2T^{(\pm)}{}_1R \tag{50}$$

due to the R-matrix block structure the diagonal element commutation rules follow

$$R_D(T_D)_1T^{(-)}{}_2 = T^{(-)}{}_2(T_D)_1R_D, \quad R_DT^{(+)}{}_2(T_D)_1 = (T_D)_1T^{(+)}{}_2R_D. \tag{51}$$

For the mutually commutative elements $A_{ii}: \ T_D = \mathrm{diag}(A_{11}, A_{22}, ...)$ one has as above

$$A_{ii}T^{(\pm)}A_{ii}^{-1} = (R_D)^{\pm 1}{}_{ii}T^{(\pm)}(R_D)^{\mp 1}{}_{ii}. \tag{52}$$

However, the diagonal block structure of R_D is different now. As a consequence the $GL_q(n|m)$ central element is the ratio of the two products corresponding to the even and odd rows

$$\mathrm{s-det}_qT = \left(\frac{\prod_{i=1}^n A_{ii}}{\prod_{k=1}^m A_{n+kn+k}}\right), \tag{53}$$

which is naturally to call by quantum superdeterminant (q-Berezinian). Cases of R-matrices depending on spectral parameters see in [58].

For the supergroup $GL_q(1|1)$ the commutation relations of the q-matrix elements $T = \left(\begin{smallmatrix} a & \beta \\ \gamma & d \end{smallmatrix}\right)$ have the form [60]-[64]

$$\begin{array}{lll} a\beta = q\beta a, & \beta d = q^{-1}d\beta, & \beta\gamma = -\gamma\beta, \\ a\gamma = q\gamma a, & \gamma d = q^{-1}d\gamma, & ad = da + \lambda\gamma\beta, \end{array} \quad \beta^2 = \gamma^2 = 0. \tag{54}$$

We use the Greek letters for odd (nilpotent) generators. The Gauss decomposition generators

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varsigma & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}, \tag{55}$$

are connected with the original ones by the formulas

$$A = a, \quad \psi = A^{-1}\beta, \quad \varsigma = \gamma A^{-1}, \quad B = d - \gamma A^{-1}\beta. \tag{56}$$

The relations

$$\begin{gathered} [A, B] = 0, \quad A\psi = q\psi A, \quad A\varsigma = q\varsigma A, \quad \psi^2 = \varsigma^2 = 0, \\ \psi\varsigma + \varsigma\psi = 0, \quad B\psi = q\psi B, \quad B\varsigma = q\varsigma B \end{gathered} \tag{57}$$

cause centrality of $GL_q(1|1)$ superdeterminant [60]

$$\mathrm{s-}\det_q T = AB^{-1} = a^2(ad - q\gamma\beta)^{-1} = a/(d - \gamma a^{-1}\beta). \tag{58}$$

In the $GL_q(2|1)$ case the q-matrix of generators has the form

$$T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} = \begin{pmatrix} M & \Psi \\ \Phi & f \end{pmatrix}. \tag{59}$$

The even M - matrix elements form $GL_q(2)$ subgroup with the commutation rules (6) and elements of each (2×2) submatrix with even generators at its diagonal form $GL_q(1|1)$ supersubgroup with (54) - type commutation relations. The other ones are read as follows

$$\begin{aligned} \alpha\beta &= -q^{-1}\beta\alpha, & c\alpha &= \alpha c, & b\gamma &= \gamma b, \\ \gamma\delta &= -q^{-1}\delta\gamma, & d\alpha &= \alpha d, & d\gamma &= \gamma d, \end{aligned}$$

$$\begin{aligned} a\beta &= \beta a + \lambda c\alpha, & b\beta &= \beta b + \lambda d\alpha, \\ a\delta &= \delta a + \lambda\gamma b, & c\delta &= \delta c + \lambda\gamma d. \end{aligned}$$

Appearing in the Gauss decomposition

$$T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \tag{60}$$

new generators have the following commutation rules

$$\begin{aligned} Ax &= qxA, & Ay &= qyA, & Az &= zA, \\ Au &= quA, & Av &= qvA, & Aw &= wA, \end{aligned}$$

$$Bx = q^{-1}xB, \qquad By = yB, \qquad Bz = qzB,$$
$$Bu = q^{-1}uB, \qquad Bv = vB, \qquad Bw = qwB,$$

$$Cx = xC, \qquad Cy = qyC, \qquad Cz = qzC,$$
$$Cu = uC, \qquad Cv = qvC, \qquad Cw = qwC,$$
$$[A, B] = [A, C] = [B, C] = 0, \qquad y^2 = z^2 = v^2 = w^2 = 0$$
$$xy = qyx, \qquad yz = -q^{-1}zy, \qquad qxz - zx = \lambda y,$$
$$uv = qvu, \qquad vw = -q^{-1}wv, \qquad uw - qw^{-1} = \lambda v,$$

$$[x, u] = [x, v] = [x, w] = 0, \quad [u, y] = [u, z] = 0,$$
$$yv + vy = 0 = yw + wy, \qquad zv + vz = 0 = zw + wz.$$

The superdeterminant

$$\text{s}-det_qT = ABC^{-1} = det_qM/C$$

is a central element. The latter expression follows from the block Gauss decomposition of (59). In particular for the $GL_q(m|n)$ matrix T in the block form one has (cf. [65])

$$s - det_qT = det_qA/det_q(D - CA^{-1}B)$$

formally the standard expression.

Generalization of the above results to the $GL_q(m|n)$ and other quantum supergroups looks rather straightforward. Although, as usual, the quantum supergroup $OSp_q(1|2)$ [56, 59] has its own peculiarities. Using notations (60), but with different grading (0, 1, 0), for the fundamental corepresentation of T [56] one has the central elements $AC = CA = B^2$ and $y = x^2/\omega, v = u^2/\omega, z \simeq x, w \simeq u, \omega = q^{1/2} - 1/q^{1/2}$. The q-matrix T has three independent generators A, x, u while in the undeformed case it is the five parameter supergroup.

6. Conclusion. In this paper we considered the Gauss decomposition of the quantum groups related to the classical Lie groups and supergroups by the elementary linear algebra and R-matrix methods. The Gauss factorization yields a new basis for these groups which is sometimes more convenient than the original one. Most of the relations among the Gauss generators are written in the $R-$matrix form. These commutation relations are simpler then the original rules. This is especially clear in the symplectic and orthogonal cases. The additional conditions completing the B, C, D - series quantum group definition, allow to extract in terms of the new generators independent ones. The number of the latter ones is equal to the dimension of the related classical group. The Gauss factorization leads naturally

to appearance of q-analogs of such classical notions as determinants, superdeterminants and minors. We also want to stress that the new basis is helpful for study the quantum group representations. In particular, it was shown in [20, 21], that it simplifies the q-bosonization problem. As we pointed out in the Introduction it looks that almost any relation and/or statement on the standard matrices are valid for the $q-$matrices being appropriately "$q-$deformed". We hope that this paper contributes to support such a feeling (cf also [70, 71, 72, 73] on the characteristic equations)[4].

Acknowledgement.The authors would like to thank L. D. Faddeev, H. Grosse, B. Jurčo, V. D. Lyakhovsky, S. Z. Pakuliak and V. O. Tarasov for helpful discussions and for pointing out important references.

The hospitality and support of the Erwin Schrödinger International Institute for Mathematical Physics during the course of this work is greatly appreciated.

The research of E.V.D. was partially sponsored by the Russian Fond of Fundamental Resarches under the grant N 94-01-01157-a and the research of P.P.K. by the DGICYT of Ministerio de Educacion y Ciencia of Spain.

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